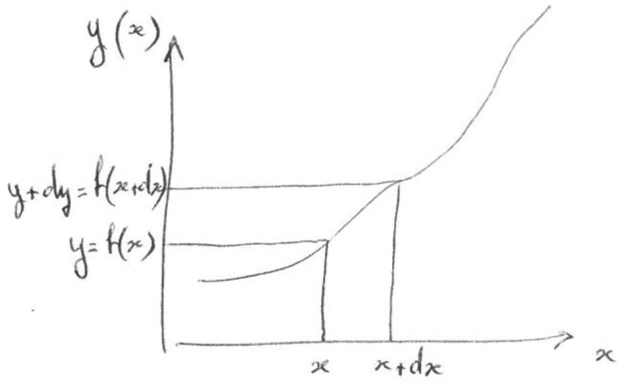
Rappels sur les dérivées

Rappels :



$$y' = \frac{dy}{dx} = \frac{f(x+dx) - f(x)}{dx}$$

= pente en x

exemple :

- $y = f(x) = x \rightarrow \cancel{y} + dy = \cancel{x} + dx$

$$\rightarrow \frac{dy}{dx} = 1 \quad \text{pente } 45^\circ$$

- $y = f(x) = x^2 \rightarrow \cancel{y} + dy = f(x+dx)^2 = (x+dx)^2$

$$= \cancel{x^2} + d^2x + 2x\,dx$$

↳ négligeable

$$= 2x\,dx$$

$$\rightarrow \frac{dy}{dx} = 2x$$

1) $y = f(x) = 1/x = x^{-1} \rightarrow \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

$$\boxed{\begin{array}{l} y = u^n \\ y' = n\,u^{n-1}u' \end{array}}$$

- $\frac{1}{x^2} \rightarrow \frac{dy}{dx} = -2x^{-3} = \frac{-2}{x^3}$
- $\frac{1}{x^3} \longrightarrow \frac{dy}{dx} = -3x^{-4} = \frac{-3}{x^4}$
- $\frac{1}{x-a} \longrightarrow \frac{dy}{dx} = -(x-a)^{-2} = \frac{-1}{(x-a)^2}$
- $\frac{1}{a-x} \longrightarrow \frac{dy}{dx} = +(a-x)^{-2} = \frac{1}{(a-x)^2}$

- $\dfrac{k}{x^{5/2}}$ $\quad y = k\,x^{-\frac{5}{2}}$

$\rightarrow \dfrac{dy}{dx} = -\dfrac{5k}{2}\, x^{-\frac{7}{2}} = \dfrac{-5k}{2x^{7/2}}$

- $V(r) = \dfrac{q}{4\pi\varepsilon r} = \dfrac{q}{4\pi\varepsilon} r^{-1}$

$\dfrac{dV(r)}{dr} = \dfrac{-q}{4\pi\varepsilon} r^{-2} = \dfrac{-q}{4\pi\varepsilon r^2}$

2) • $\dfrac{1}{x^2+a^2}$ $\rightarrow y = (x^2+a^2)^{-1}$

$\rightarrow \dfrac{dy}{dx} = -(x^2+a^2)^{-2} \cdot 2x = \dfrac{-2x}{(x^2+a^2)^2}$

- $\dfrac{1}{(x^2+a^2)^{3/2}}$ $\rightarrow y = (x^2+a^2)^{-3/2}$

$\rightarrow \dfrac{dy}{dx} = -\dfrac{3}{\cancel{2}} \cdot (x^2+a^2)^{-\frac{5}{2}} \cdot \cancel{2}x = \dfrac{-3x}{(x^2+a^2)^{5/2}}$

- $\dfrac{1}{(a^2-x^2)^{5/2}}$ $\rightarrow y = (a^2-x^2)^{-5/2}$

$\dfrac{dy}{dx} = \cancel{-}\dfrac{5}{\cancel{2}} (a^2-x^2)^{-\frac{7}{2}} \cdot \cancel{-2}x = \dfrac{5x}{(a^2-x^2)^{7/2}}$

- $E(z) = \dfrac{az}{(z^2+a^2)^{3/2}} = az \cdot (z^2+a^2)^{-3/2}$

$\boxed{(uv)' = u'v + uv'}$

$\dfrac{dE(z)}{dz} = a \cdot (z^2+a^2)^{-3/2} + az \cdot -\dfrac{3}{\cancel{2}} (z^2+a^2)^{-5/2} \cdot \cancel{2}z$

$= a(z^2+a^2)^{-3/2} + -3az^2 (z^2+a^2)^{-5/2}$

en passant par $\dfrac{u}{v}$: $= (z^2+a^2)^{-\frac{5}{2}} (-2az^2 + a^3)$

3) • $y(x) = (2-x)^4 \rightarrow \dfrac{dy}{dx} = 4(2-x)^3 \cdot -1$

$= -4(2-x)^3$

• $y(x) = (2+x^2)^3 \rightarrow \dfrac{dy}{dx} = 3(2+x^2) \cdot 2x = 6x(2+x^2)$

• $x(y) = (3y^2+1)^2 \rightarrow \dfrac{dx}{dy} = 2(3y^2+1) \cdot 6y = 12y(3y^2+1)$

4) Dérivée d'un produit : $y = uv \rightarrow y' = u'v + uv'$

• $3x^3(2-4x^4)^5$ $\quad u = 3x^3 \quad u' = 3 \cdot 3x^2 \quad v = (2-4x^4)^5 \quad v' = 5 \cdot (2-4x^4)^4 \cdot -4 \cdot 4x^3$

$= -80x^3(2-4x^4)^4$

$\rightarrow \dfrac{dy}{dx} = 3 \cdot 3x^2(2-4x^4)^5 + 3x^3 \cdot -80x^3(2-4x^4)^4$

$= 9x^2(2-4x^4)^5 - 240x^6(2-4x^4)^4$

• $\dfrac{x}{1+x^2} \rightarrow y(x) = x(1+x^2)^{-1}$

$\rightarrow \dfrac{dy}{dx} = (1+x^2)^{-1} + -x(1+x^2)^{-2} \cdot 2x$

$= \dfrac{1}{1+x^2} - \dfrac{2x^2}{(1+x^2)^2} = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

• $y(x) = \dfrac{(1-x^2)^3}{(1+4x^3)^2} = (1-x^2)^3(1+4x^3)^{-2}$

$\dfrac{dy}{dx} = 3(1-x^2)^2 \cdot -2x \cdot (1+4x^3)^{-2} + (1-x^2)^3 \cdot -2(1+4x^3)^{-3} \cdot 12x^2$

$= \dfrac{-6x(1-x^2)^2}{(1+4x^3)^2} - \dfrac{24x^2(1-x^2)^3}{(1+4x^3)^3}$

• $y(x) = x\left(1+\dfrac{2}{1-x}\right) = x + \dfrac{2x}{1-x}$ $\qquad y = u+v \rightarrow du = du + dv$

$1 + \dfrac{2(1-x)+2x}{(1-x)^2}$

$\dfrac{dy}{dx} =$

$= 1 + \dfrac{2}{(1-x)^2}$

$=$

• $y(x) = \sqrt{1-\dfrac{ax}{x^2+a^2}} = \left(1-\dfrac{ax}{x^2+a^2}\right)^{1/2}$

$ax^2 - a^3$

$\dfrac{dy}{dx} = \dfrac{1}{2}\left(1-\dfrac{ax}{x^2+a^2}\right)^{-1/2} u'$ $\qquad \left(u = \dfrac{-ax}{x^2+a^2} = -ax(x^2+a^2)^{-1} \right.$

$u' = -a(x^2+a^2)^{-1} - ax(x^2+a^2)^{-2} \cdot 2x$

$= \dfrac{-a}{2(x^2+a^2)\left(1-\frac{ax}{x^2+a^2}\right)^{1/2}} + \dfrac{ax^2}{(x^2+a^2)^2\left(1-\frac{ax}{x^2+a^2}\right)^{1/2}} = \dfrac{2ax^2 - a(x^2+a^2)}{2(x^2+a^2)^2\sqrt{}}$

5)

$$\boxed{\begin{array}{ll} y = \cos u & y' = -u' \sin u \\ y = \sin u & y' = u' \cos u \\ y = \tan u & y' = u'(1+\tan^2 u) = \dfrac{u'}{\cos^2 u} \end{array}}$$

$$\boxed{\begin{array}{l} \cos(a+b) = \cos a \cos b - \sin a \sin b \\ \sin(a+b) = \cos a \sin b + \sin a \cos b \end{array}}$$

Démonstration : $\bullet\ y = \cos x$: $\cancel{y} + dy = \cos(x + dx)$

$$= \cancel{\cos x} \underbrace{\cos dx}_{\approx 1} - \sin x \underbrace{\sin dx}_{\approx dx}$$

$$\boxed{\left(\begin{array}{l} \cos \varepsilon = 1 \\ \sin \varepsilon = \varepsilon \end{array} \right)}$$

$$dy = - dx \sin x$$

$$\hookrightarrow \frac{dy}{dx} = - \sin x$$

$\bullet\ y = \sin x$: $\cancel{y} + dy = \sin(x + dx)$

$$= \cos x \times \underbrace{\sin dx}_{\approx dx} + \cancel{\sin x} \underbrace{\cos dx}_{\approx 1}$$

$$dy = dx \cos x$$

$$\rightarrow \frac{dy}{dx} = \cos x$$

$\bullet\ y = \tan x = \dfrac{\sin x}{\cos x} \rightarrow y' = \dfrac{\cos x \cdot \cos x - \sin x \cdot - \sin x}{\cos^2 x}$

$$= \frac{1}{\cos^2 x}$$

$\bullet\ y = \sin 2x$

$$\boxed{\left(\sin 2a = 2 \sin a \cos a\right) \left(\cos 2a = \cos^2 a - \sin^2 a\right)}$$

$= 2 \sin x \cos x \rightarrow \dfrac{dy}{dx} = 2(\cos x \cos x - \sin x \sin x)$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= 2 \cos 2x$$

ou bien : $\boxed{(g \circ f)' = f' \times g'(f)}$

$$\sin 2x \rightarrow \frac{dy}{dx} = 2 \cdot \cos 2x$$

$$\boxed{\begin{array}{l} (g \circ f)' = f' \times (g' \circ f) \\ (g(f))' = f' \times g'(f) \end{array}}$$

- $y(x) = \sin\frac{x}{2} \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

- $y(x) = \cos^2\left(\frac{x}{2}\right) \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot -2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$$u = \frac{x}{2} \rightarrow u' = \frac{1}{2}$$
$$v = \cos^2(\ ) \rightarrow v' = -2\cos(\ )\sin\left(\frac{x}{2}\right)$$

$(\sin 2a = 2\sin a \cos a)$

$$\rightarrow \frac{dy}{dx} = -\sin\frac{x}{2}\cos\frac{x}{2} = -\frac{1}{2}\sin x$$

- $y(x) = \sin^2\left(\frac{x}{2}\right) \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \cancel{2}\sin\frac{x}{2}\cos\frac{x}{2} = \frac{1}{2}\sin x$

- $y(x) = \cos^2\left(\frac{x^2}{2}\right) \quad \rightarrow \quad \frac{dy}{dx} = x \cdot -2\sin\left(\frac{x^2}{2}\right)\cos\left(\frac{x^2}{2}\right) = -x\sin(x^2)$

- $y(x) = \sin^2\left(\frac{x^2}{2}\right) = \left(\sin\frac{x^2}{2}\right)^2$

$$\rightarrow \frac{dy}{dx} = 2\sin\frac{x^2}{2} \cdot x\cos\frac{x^2}{2} = 2x\sin\frac{x^2}{2}\cos\frac{x^2}{2}$$
$$= x\sin(x^2)$$

- $y(x) = \tan(x^3) = \frac{\sin x^3}{\cos x^3}$

$$\rightarrow \frac{dy}{dx} = \frac{3x^2}{\cos^2(x^3)}$$

- $\cos(x^3) \quad \rightarrow \quad \frac{dy}{dx} = -3x^2\sin x^3$

- $y = \sin^2 x\cos x \quad \rightarrow \quad \frac{dy}{dx} = 2\cos x\cos x\overset{\sin x}{\vee} + \sin^2 x \cdot -\sin x$

$$= 2\cos^2 x\overset{\sin x}{\vee} - \sin^3 x$$

- $y = \sqrt{(1-\cos 3x)^3} = (1-\cos 3x)^{3/2}$

$$\rightarrow \frac{dy}{dx} = \frac{3}{2}(1-\cos 3x)^{1/2}\left(-3(-\sin 3x)\right)$$
$$= \frac{9}{2}(1-\cos 3x)^{1/2}\sin 3x$$

- $y = \sin^2(4x^2-1) \quad \rightarrow \quad \frac{dy}{dx} = 2\sin(4x^2-1) \cdot 8x\cos(4x^2-1)$

$$= 16x\sin(4x^2-1)\cos(4x^2-1)$$
$$= 8x\sin\left(2(4x^2-1)\right)$$

$\sin 2a = 2\sin a\cos a$

- $$n(A) = \frac{\sin\left(\frac{A+D_m}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$ $\sin(a-b) = \sin a \cos b - \sin b \cos a$

b a

$$\frac{dn(A)}{dA} = \frac{\frac{1}{2}\cos\left(\frac{A+D_m}{2}\right)\sin\frac{A}{2} - \frac{1}{2}\sin\left(\frac{A+D_m}{2}\right)\cos\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\sin\left(\frac{A}{2}-\frac{A+D_m}{2}\right)}{\sin^2\left(\frac{A}{2}\right)} = \frac{\sin\left(-\frac{D_m}{2}\right)}{2\sin^2\left(\frac{A}{2}\right)}$$

6)

$$\boxed{\begin{array}{ll} y = e^u & y' = u' e^u \\ y = \ln|u| & y' = \frac{u'}{u} \end{array}}$$

- $$y(x) = \ln(1-x)^2 = 2\ln(1-x) \rightarrow \frac{dy}{dx} = \frac{-2}{1-x}$$

- $$y(x) = \ln\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$$

$$\rightarrow \frac{dy}{dx} = \left(\frac{1/2}{\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right) \Big/ \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$= \frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$

$$= \frac{1}{\sin\left(\frac{\pi}{2}+x\right)}$$

$$= \frac{1}{\underbrace{\sin\frac{\pi}{2}}_{1}\cos x + \sin x \underbrace{\cos\frac{\pi}{2}}_{0}} = \frac{1}{\cos x}$$

- $$y(x) = e^{-x^2}(3x^2+6x-1)$$

$$\rightarrow \frac{dy}{dx} = -2x e^{-x^2}(3x^2+6x-1) + e^{-x^2}(6x+6)$$

$$= e^{-x^2}(-6x^3-12x^2+8x+6)$$

$$= 2e^{-x^2}(-3x^3-6x^2+4x+3)$$

- $y(x) = \frac{1}{\sin x} = (\sin x)^{-1} \rightarrow \frac{dy}{dx} = -\sin^{-2} x \cos x$

- $V(x) = \frac{\lambda}{4\pi\varepsilon} \ln(1-x)$

$$\rightarrow \frac{dV(x)}{dx} = \frac{\lambda}{4\pi\varepsilon} \cdot \frac{-1}{(1-x)} = \frac{-\lambda}{4\pi\varepsilon\ (1-x)}$$

7) Fonctions réciproques

Soit une fcθ de x : $y = f(x)$

Sa fcθ réciproque est : $x = g(y)$

Leurs dérivés sont liées entres elle :

$$\left.\begin{array}{l} y = f(x) \rightarrow y' = \frac{dy}{dx} = f' \\ x = g(y) \rightarrow x' = \frac{dx}{dy} = g' \rightarrow dy = \frac{dx}{x'} \end{array}\right\} \boxed{y' = \frac{1}{x'}}$$

$f' = \frac{1}{g'}$

- $y = \cos x$: réciproque : $x = \arccos y$

ex : $\begin{cases} \arccos \frac{1}{2} = \frac{\pi}{3} \\ \arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4} \end{cases}$

- $y = \arccos x$

$y = \arccos x$, réciproque $x = \cos y \rightarrow x' = -\sin y$

$$\rightarrow y' = \frac{-1}{\sin y}$$

Exprimer y en fcθ de x :

$$\sin^2 y + \cos^2 y = 1$$

$$\rightarrow \sin^2 y = 1 - x^2 \rightarrow \sin y = \sqrt{1-x^2}$$

$$\rightarrow y' = \frac{-1}{\sqrt{1-x^2}}$$

$1 - x^2 > 0$

$1 > x^2$

$-1 < x < 1$

domaine de déf

- $y(x) = \arcsin x$

réciproque : $x = \sin y \rightarrow x' = \cos y \rightarrow y' = \frac{1}{\cos y}$

$$\cos^2 y = 1 - x^2 \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

- $y(x) = \arctan x$

$$x = \tan y \rightarrow x' = \frac{1}{\cos^2 y} \rightarrow y' = \cos^2 y$$

$$\cos^2 y = 1 - \sin^2 y \rightarrow$$
$$= 1 - \tan^2 y \cos^2 y$$
$$\rightarrow \cos^2 y \left(1 + x^2\right) = 1 \rightarrow \cos^2 y = \frac{1}{1+x^2}$$
$$\rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

- $y(x) = \ln x$

$$x = e^y \rightarrow x' = e^y = x \rightarrow y' = \frac{1}{x'} = \frac{1}{x}$$

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1) a):

- $y(x) = \left(x^2+1\right)^2 \rightarrow \dfrac{dy}{dx} = 2\left(x^2+1\right) \cdot 2x = 4x\left(x^2+1\right)$
- $y(t) = \cos^3(\omega t) \rightarrow \dfrac{dy}{dt} = 3\cos^2(\omega t) \cdot -\omega \sin \omega t = -3\omega \cos^2(\omega t)\sin(\omega t)$
- $y(x) = \ln\left(2x^2\right) \rightarrow \dfrac{dy}{dx} = \dfrac{4x}{2x^2} = \dfrac{2}{x}$
- $y(t) = gt^2 + at + 1 \rightarrow \dfrac{dy}{dt} = 2gt + a$
- $f(t) = t^2\left(3t^2 + 2t + 1\right) \rightarrow \dfrac{df(t)}{dt} = 2t\left(3t^2 + 2t + 1\right) + t^2(6t + 2)$

$$= 6t^3 + 4t^2 + 2t + 6t^3 + 2t^2$$
$$= 12t^3 + 6t^2 + 2t$$

- $y(x) = \dfrac{bx}{b^2+x^2} = bx\left(b^2+x^2\right)^{-1}$

$$\rightarrow \frac{dy}{dx} = b\left(b^2+x^2\right)^{-1} + bx \cdot -1\left(b^2+x^2\right)^{-2} 2x$$
$$= \frac{b}{b^2+x^2} - \frac{2bx^2}{\left(b^2+x^2\right)^2} = \frac{b^4 - bx^2}{\left(b^2+x^2\right)^2}$$

- $V(x,y) = \dfrac{k}{\sqrt{x^2+y^2}} = k\,(x^2+y^2)^{-\frac{1}{2}}$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$ Diff. totale exacte

$$\frac{\partial V}{\partial x} = -\frac{k}{2}(x^2+y^2)^{-\frac{3}{2}}.\,2x = -xk\,/(x^2+y^2)^{3/2}$$

$$\frac{\partial V}{\partial y} = -yk\,/\,(x^2+y^2)^{3/2}$$

- $\gamma(h,h') = \dfrac{h}{h-h'} = h\,(h-h')^{-1}$

$$d\gamma = \frac{\partial \gamma}{\partial h}dh + \frac{\partial \gamma}{\partial h'}dh'$$

$$\frac{\partial \gamma}{\partial h} = (h-h')^{-1} + h.\bar{1}(h-h')^{\frac{-2}{1}} = \frac{1}{h-h'} - \frac{-h}{(h-h')^2} = \frac{h-h'+h}{(h-h')^2} = \frac{2h-h'}{(h-h')^2}$$

$$\frac{\partial \gamma}{\partial h'} = -h\,(h-h')^{-2}.-1 = h/(h-h')^2$$

- $R = \rho\,\dfrac{\theta_2-\theta_1}{\theta_1} = \rho\,(\theta_2-\theta_1)\,\theta_1^{-1}$ si ρ variable

$$dR = \frac{\partial R}{\partial \theta_1}d\theta_1 + \frac{\partial R}{\partial \theta_2}d\theta_2 + \frac{\partial R}{\partial \rho}d\rho$$

$$\frac{\partial R}{\partial \theta_1} = \rho\left(-1\,\theta_1^{-1} + -1.(\theta_2-\theta_1)\,\theta_1^{-2}\right) = \rho\left(-\cancel{\theta_1^{-1}} - (\theta_2-\cancel{\theta_1})\,\theta_1^{-2}\right)$$

$$= \cancel{\frac{-\rho}{\theta_1}} - \frac{\rho(\theta_2-\cancel{\theta_1})}{\theta_1^2} = -\frac{\rho\,\theta_2}{\theta_1^2}$$

$$\frac{\partial R}{\partial \theta_2} = \frac{\rho}{\theta_1} \qquad \frac{\partial R}{\partial \rho} = \frac{\theta_2-\theta_1}{\theta_1}$$

1 b) Différentielle logarithmique défi$^\circ$: $d\ln y = \dfrac{dy}{y}$

- $y(x) = (x^2+1)^2$

$$\to \ln y(x) = 2\ln(x^2+1)$$

$$\frac{d\ln y(x)}{dx} = \frac{4x}{x^2+1} = \frac{dy}{y\,dx}$$

ou $\dfrac{dy}{y} = \dfrac{4x}{x^2+1}dx$