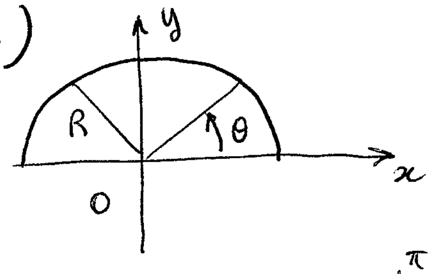
Géométrie de Masses

I 1)



$m\,\overrightarrow{OG} = \int \overrightarrow{OM}\,dm$

symétrie 1/1 Oy $\rightarrow$ $G \in Oy$: $\boxed{x_G = 0}$

$m\,y_G = \int y\,dm$ avec $\left| \begin{array}{l} dm = \lambda \underbrace{R\,d\theta}_{dl} \\ y = R\sin\theta \end{array} \right.$

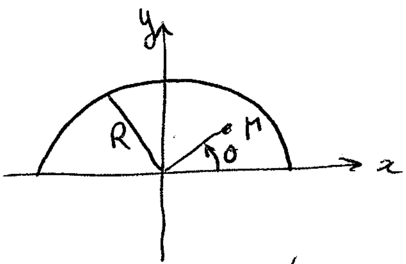
$\rightarrow \quad m y_G = \lambda R^2 \int_0^{\pi} \sin\theta\,d\theta = 2\lambda R^2$

$m = \pi R \lambda \quad \rightarrow \quad \boxed{y_G = \frac{2R}{\pi}}$

On peut aussi appliquer le théorème de Guldin. En tournant 1/1 Ox (traverse l'axe Oy), l'arc de cercle décrit une sphère de surface $4\pi R^2$:

$S = 4\pi R^2 = 2\pi L\, y_G = 2\pi . \pi R\, y_G \quad \rightarrow \quad y_G = \frac{4\pi R^2}{2\pi^2 R} = \frac{2R}{\pi}$

2)



de même, par symétrie : $\boxed{x_G = 0}$

$m\,y_G = \int y\,dm$ avec $\left| \begin{array}{l} dm = \sigma \underbrace{r\,dr\,d\theta}_{dS \text{ polaire}} \\ y = r\sin\theta \end{array} \right.$

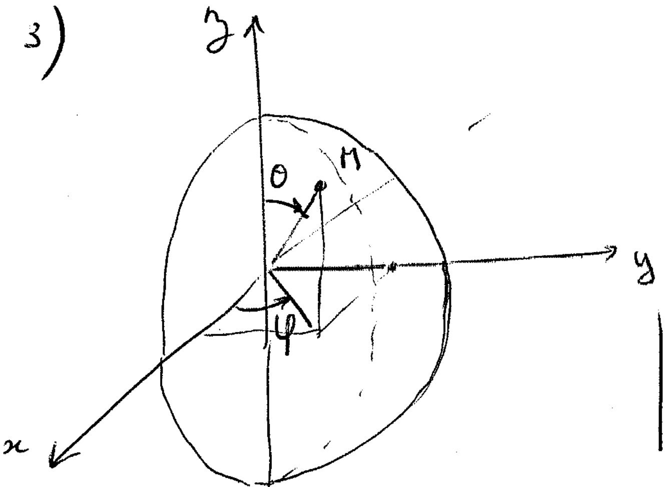
$\rightarrow \quad m\,y_G = \int \sigma r^2 dr \sin\theta\,d\theta = \sigma \int_0^R r^2 dr \int_0^{\pi} \sin\theta\,d\theta = \sigma \frac{R^3}{3} . 2$

avec $m = \sigma \frac{\pi R^2}{2} \quad \rightarrow \quad \boxed{y_G = \frac{4R}{3\pi}}$

par Guldin : $V = 2\pi S\, y_G \quad \rightarrow \quad \frac{4}{3}\pi R^3 = 2\pi \frac{\pi R^2}{2} y_G$

$\rightarrow \quad y_G = \frac{4R}{3\pi}$

3)



Oy est axe de symétrie : $\boxed{\begin{array}{l} x_G = 0 \\ z_G = 0 \end{array}}$

$m\,y_G = \int y\,dm$ avec

$\left| \begin{array}{l} dm = \rho\,dV = \rho\, r^2 \sin\theta\,dr\,d\theta\,d\varphi \\ y = r\sin\theta\sin\varphi \end{array} \right.$

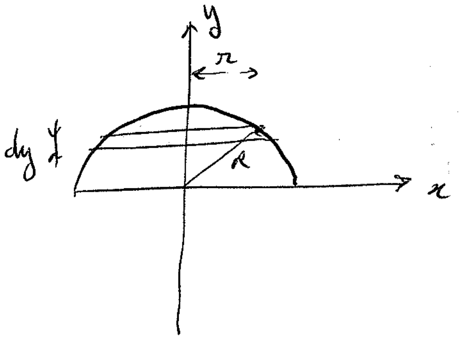
$$\rightarrow \quad m\, y_G = \rho \int_0^R \int_0^\pi \int_0^\pi r^3 \sin^2\theta \, d\theta \, dr \, \sin\varphi \, d\varphi$$

$$= \rho \int_0^R r^3 dr \underbrace{\int_0^\pi \sin^2\theta \, d\theta}_{\frac{1}{2}\left[\theta - \cos 2\theta\right]_0^\pi} \underbrace{\int_0^\pi \sin\varphi \, d\varphi}_{2}$$

$$= \rho \pi \frac{R^4}{4}$$

avec $m = \rho . \frac{2}{3}\pi R^3 \rightarrow \boxed{y_G = \frac{3R}{8}}$

On peut aussi considérer la ½ sphère comme un empilement de disques d'épaisseur dy, de rayon r variable avec y. Pour chaque tranche, on connait la posi° du centre de masse : $y_{G_i} = y$



$$m\, y_G = \int y_{G_i} \, dm_i \quad \text{avec} \quad \left| \begin{array}{l} dm_i = \rho \pi r^2 dy \\ r^2 = f(y) \\ \quad = R^2 - y^2 \end{array} \right.$$

$$\rightarrow \quad m\, y_G = \pi \rho \int y\, r^2 dy$$

$$= \pi \rho \int_0^R y (R^2 - y^2) \, dy$$

$$= \pi \rho R^2 \frac{R^2}{2} - \pi \rho \frac{R^4}{4} = \pi \rho \frac{R^4}{4}$$

$m = \rho . \frac{2}{3}\pi R^3 \implies y_G = \frac{3R}{8}$

Galdin

$V = 2\pi S y_G$

$\frac{4}{3}\pi R^3 = 2\pi \, 2\pi R^2 y_G$

4) S_1 : cylindre plein de rayon r et hauteur h
S_2 : cylindre (R, H) évidé
S : cylindre plein, rayon R, hauteur H

$S = S_1 + S_2$

$$m\,\vec{OG} = m_1\,\vec{OG_1} + m_2\,\vec{OG_2}$$

$$\rightarrow \quad m_2\,\vec{OG_2} = m\,\vec{OG} - m_1\,\vec{OG_1}$$

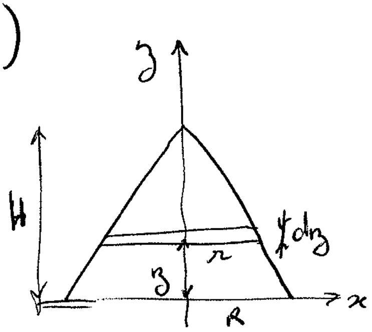
$$\vec{OG} = \frac{H}{2}\vec{e_z} \qquad \vec{OG_1} = \frac{h}{2}\vec{e_z} \qquad m = \rho\pi R^2 H \qquad m_1 = \rho\pi r^2 h$$

$$\rightarrow \quad m_2\,\vec{OG_2} = \rho\pi\left(R^2\frac{H^2}{2} - r^2\frac{h^2}{2}\right)\vec{e_z}$$

et $\quad m_2 = m - m_1 = \rho\pi\left(R^2 H - r^2 h\right)$

$$\rightarrow \quad \boxed{\vec{OG_2} = \frac{1}{2}\left(\frac{R^2H^2 - r^2h^2}{R^2H - r^2h}\right)\vec{e_z}}$$

5)



axe de symétrie Oz : $\boxed{\begin{array}{l} x_G = 0 \\ y_G = 0 \end{array}}$

On peut décomposer le cône en disques d'épaisseur dz.

Pour chaque disque : $m z_G = \int z_{Gi}\, dm_i$ avec $z_{Gi} = z$

$dm_i = \underbrace{\rho\pi r^2 dz}_{dV}$

$$\rightarrow \quad m z_G = \pi\rho \int z\, r^2 dz \qquad \text{et} \quad r = f(z)$$

$$\frac{r}{R} = \frac{H-z}{H} \quad \rightarrow \quad r = \frac{R}{H}(H-z)$$

$$\rightarrow \quad m z_G = \pi\rho\frac{R^2}{H^2}\int_0^H z\,(H-z)^2 dz$$

$$= \pi\rho\frac{R^2}{H^2}\left[H^2\frac{z^2}{2} + \frac{z^4}{4} - 2H\frac{z^3}{3}\right]_0^H$$

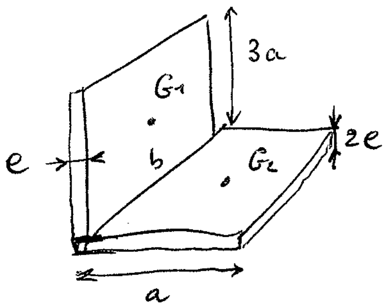
$$= \pi\rho\frac{R^2}{H^2}\left(\frac{H^4}{2} + \frac{H^4}{4} - 2\frac{H^4}{3}\right) = \frac{\rho\pi R^2 H^2}{12}$$

$$m = \int dm = \int \rho\pi r^2 dz = \frac{R^2}{H^2}\int_0^H \rho\pi\,(H-z)^2 dz$$

$$= \frac{R^2}{H^2}\rho\pi\left[H^2 z + \frac{z^3}{3} - 2H\frac{z^2}{2}\right]_0^H = \frac{\rho\pi R^2 H}{3}$$

$$\rightarrow \quad \boxed{z_G = \frac{H}{4}}$$

6)



$$m\,\overrightarrow{G_1G} = m_1\,\overrightarrow{G_1G_1} + m_2\,\overrightarrow{G_1G_2} = m_2\,\overrightarrow{G_1G_2}$$

$$\left.\begin{array}{l} m_1 = \rho\,.\,3a\,.\,b\,.\,e \\ m_2 = \rho\,.\,a\,.\,b\,.\,2e \end{array}\right\} \quad m = m_1 + m_2 = 5abe\rho$$

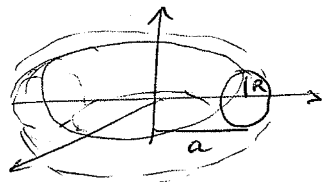
$$\rightarrow \boxed{\overrightarrow{G_1G} = \frac{2}{5}\,\overrightarrow{G_1G_2}}$$

II 1) $S = 2\pi L\, x_G(\mathcal{E})$ $\qquad L = \text{périmètre} = 2\pi R \qquad x_G(\mathcal{E}) = a$

$$\rightarrow \boxed{S = 4\pi^2 R a}$$

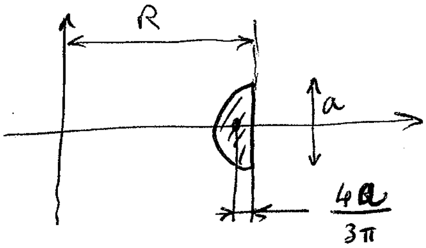
$V = 2\pi S\, x_G(\Sigma)$ $\qquad S = \text{surface du cercle} = \pi R^2 \qquad x_G(\Sigma) = a$

$$\rightarrow \boxed{V = 2\pi^2 R^2 a}$$



$S = 2\pi R\,.\,2\pi a$
$V = \pi R^2\,.\,2\pi a$

2) Volume du cylindre entier : $\pi R^2 h$
Volume du cylindre intérieur : $\pi r^2 h$
Volume du $\frac{1}{2}$ tore : ⚠ ce n'est pas la moitié du volume du tore, car il y a + de volume à l'extérieur qu'à l'intérieur si on coupe le tore sur sur plan médiateur.



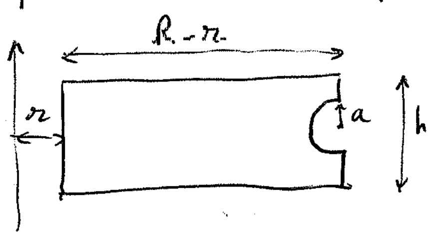
$\frac{4a}{3\pi}$ d'après I 2)

$$V = 2\pi\, \underset{\sim}{S}\, x_G$$

$$\rightarrow V = 2\pi\,.\,\frac{\pi a^2}{2}\,.\left(R - \frac{4a}{3\pi}\right)$$

$$\rightarrow \boxed{V = \pi R^2 h - \pi r^2 h - \pi^2 a^2\left(R - \frac{4a}{3\pi}\right)}$$

On peut tout faire grâce au théorème de Guldin :



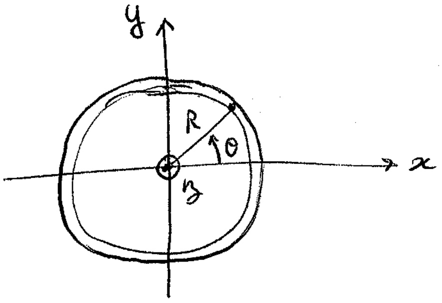
$$V = 2\pi\, S\, x_G$$

$m\,\vec{OG} = m_2\,\vec{OG_2} - m_1\,\vec{OG_1}$ (2) : rectangle
(1) : cercle

$$S x_G = S_2 x_2 - S_1 x_1$$
$$= (R-r)h \cdot \underbrace{\left(r + \frac{R-r}{2}\right)}_{\frac{r}{2}+\frac{R}{2}} - \frac{\pi a^2}{2}\left(R - \frac{4a}{3\pi}\right)$$
$$= \cancel{\frac{Rhr}{2}} - \frac{r^2}{2}h + \frac{R^2 h}{2} - \cancel{\frac{rhR}{2}}$$
$$= \frac{h}{2}\left(R^2 - r^2\right) - \frac{\pi a^2}{2}\left(R - \frac{4a}{3\pi}\right)$$

$\rightarrow$ $\boxed{V = \pi h\left(R^2 - r^2\right) - \pi^2 a^2\left(R - \frac{4a}{3\pi}\right)}$

III 1)



$I_{Oz} = \int d^2\,dm$ avec $d = R$
$dm = R\,d\theta\,\lambda$

$I_{Oz} = \lambda \int R^2 . R\,d\theta = \lambda R^3 . 2\pi$

$m = \lambda . 2\pi R \rightarrow \lambda = \frac{m}{2\pi R}$

$\rightarrow$ $\boxed{I_{Oz} = m R^2}$

$I_{Oy} = I_{Oxy} + I_{Oyz}$ $\quad I_{Oxy} = 0$ $\left(= \int z^2\,dm \right) = \underset{0}{z^2} \int dm \Big)$

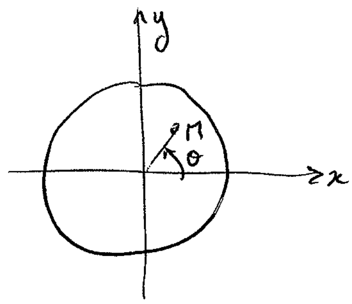
$\rightarrow I_{Oy} = I_{Oyz}$

et $I_{Oz} = I_{Oxz} + I_{Oyz} = 2\,I_{Oyz}$ par symétrie.

$\rightarrow I_{Oy} = \frac{I_{Oz}}{2} = \frac{mR^2}{2}$

$\boxed{I_{Oy} = I_{Ox} = \frac{mR^2}{2}}$

2)



$$I_{Oz} = \int d^2\, dm \qquad \left|\begin{array}{l} d = r \\ dm = \sigma\, r\, d\theta\, dr \end{array}\right.$$

$$I_{Oz} = \sigma \int r^3 d\theta\, dr = \sigma \int_0^R r^3 dr \int_0^{2\pi} d\theta = \frac{\sigma r^4}{4} . 2\pi$$

$$= \frac{\pi \sigma R^4}{2}$$

$$m = \sigma . \pi R^2 \quad \rightarrow \quad \boxed{I_{Oz} = \frac{m R^2}{2}}$$

$$I_{Oz} = 2\, I_{Oyz} \quad \text{et} \quad I_{Oy} = \cancel{I_{Oxy}} + I_{Oyz} = I_{Oyz} \rightarrow I_{Oy} = \frac{I_{Oz}}{2}$$

$$\rightarrow \boxed{I_{Oy} = \frac{m R^2}{4}}$$

calcul :

$$I_{Oy} = \int_0^R \int_0^{2\pi} (r \sin\theta)^2 . \sigma\, r\, dr\, d\theta = \sigma \int_0^R r^3 dr \underbrace{\int_0^{2\pi} \sin^2\theta\, d\theta}$$

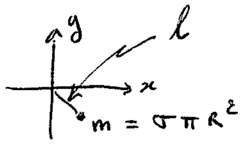
$$\frac{1}{2}\left(2\pi - 0 - \cancel{1} + \cancel{1}\right) = \underbrace{\left[\frac{\theta - \cos 2\theta}{2}\right]_0^{2\pi}}_{\pi}$$

$$= \sigma \frac{R^4}{4} \pi = \frac{I_{Oz}}{2}$$

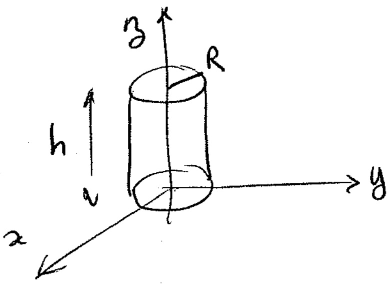
Rayon de giraθ / Oz :

$$\left.\begin{array}{l} m = \sigma . \pi R^2 \\ I_{Oz} = \dfrac{m R^2}{2} \end{array}\right\} \quad I_{Oz} = \ell^2 m \rightarrow \ell = \sqrt{\frac{I_{Oz}}{m}} = \frac{R}{\sqrt{2}}$$

$$\boxed{\ell = \frac{R}{\sqrt{2}}} \quad \equiv$$



3)



$$I_{Oz} = \int (x^2 + y^2)\, dm \qquad dm = \underbrace{\rho\, r\, dr\, d\theta\, dz}_{dV}$$

$$x^2 + y^2 = r^2$$

$$I_{Oz} = \rho \int r^3 dr\, d\theta\, dz$$

$$\rightarrow \quad I_{Oz} = \rho \int_0^R r^3 dr \int_0^{2\pi} d\theta \int_0^h dz = \rho \frac{R^4}{4} . 2\pi . h$$

$$m = \pi R^2 . h . \rho \quad \rightarrow \quad \boxed{I_{Oz} = \frac{m R^2}{2}}$$

Autre solut° : On peut considérer le cylindre c̄ 1 empilement de disques d'épaisseurs dz et de $dI = \frac{R^2}{2} dm$

$$I = \int dI = \int dm \frac{R^2}{2} = \frac{1}{2} R^2 \int dm = \frac{1}{2} m R^2$$

4)

$$I = I_2 - I_1 \qquad I_2 = \frac{1}{2} m_2 R_2^2 \qquad m_2 = \rho \pi R_2^2 h$$

$$I_1 = \frac{1}{2} m_1 R_1^2 \qquad m_1 = \rho \pi R_1^2 h$$

$$\rightarrow \quad I = \frac{1}{2} \rho \pi h \left(R_2^4 - R_1^4\right)$$

$$m = m_2 - m_1 = \rho \pi h \left(R_2^2 - R_1^2\right) \rightarrow \rho = \frac{m}{\pi h \left(R_2^2 - R_1^2\right)}$$

$$\rightarrow \boxed{I = \frac{1}{2} m \frac{R_2^4 - R_1^4}{R_2^2 - R_1^2}} = \frac{m}{2} \left[R_2^2 + R_1^2\right]$$

5)

$$\left.\begin{aligned} d(M, Oz) &= r \sin\theta \\ dm &= \rho r^2 \sin\theta \, d\theta \, d\varphi \, dr \end{aligned}\right\} \quad I_{Oz} = \rho \int_0^R \int_0^\pi \int_0^{2\pi} (r \sin\theta)^2 r^2 \sin\theta \, d\theta \, dr \, d\varphi$$

$$I_{Oz} = \rho \underbrace{\int_0^R r^4 dr}_{\frac{R^5}{5}} \int_0^\pi \sin^3\theta \, d\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin\theta \left(1 - \cos^2\theta\right) d\theta = \underbrace{\int_0^\pi \sin\theta \, d\theta}_{2} - \int_0^\pi \sin\theta \cos^2\theta \, d\theta$$

$$\int_0^\pi \overbrace{\sin\theta \underbrace{\cos^2\theta}_{u^n} d\theta}^{-du} = \frac{1}{3}\left[-\cos^3\theta\right]_0^\pi = -\frac{1}{3}(-1-1) = 2/3$$

$d(u^n) = n u^{n-1} du$

$d(\cos^2\theta) = 2(-\sin\theta)\cos\theta$

$\int u^n du = \frac{u^{n+1}}{n+1}$

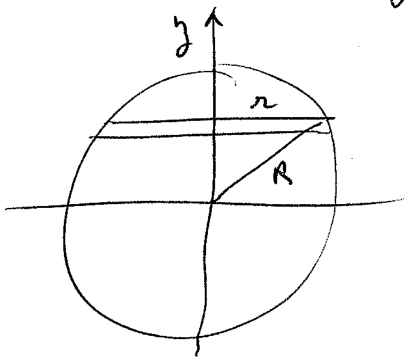
$$\rightarrow I_{Oz} = \rho \cdot \frac{R^5}{5} \cdot 2\pi \cdot \frac{4}{3}$$

$$\boxed{I_{Oz} = \frac{8}{15} \rho \pi R^5}$$

avec $m = \frac{4}{3}\pi R^3 \rho$ $\rightarrow$ $\boxed{I_{Oz} = \frac{2}{5} m R^2}$

autre méthode : sphère = empilement de disques d'épaisseur dz, de rayon $r = f(z)$: $r^2 = R^2 - z^2$

$$dm = \rho \pi (R^2 - z^2) dz$$



moment d'inertie d'1 disque : $dI = \frac{1}{2} dm\, r^2$

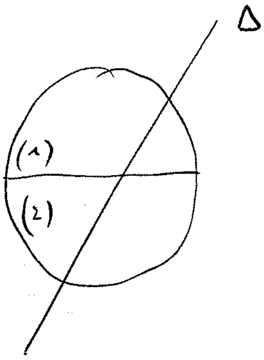
$$dI = \frac{1}{2}(R^2 - z^2) . \rho\pi (R^2 - z^2) dz$$

$$= \frac{1}{2}\rho\pi (R^2 - z^2)^2 dz$$

$$I_{Oz} = \int_{-R}^{R} dI = \frac{1}{2}\rho\pi \int_{-R}^{R} \underbrace{(R^2 - z^2)^2}_{R^4 - R^2z^2 + z^4 - R^2z^2} dz$$

$$= \frac{1}{2}\rho\pi \left[2\frac{R^5}{5} + R^4 . 2R - 2R^2 \frac{\cancel{R}^3}{3} . 2 \right] = \frac{8}{15}\rho\pi R^5$$

6)



par symétrie : $I_{\Delta 1} = I_{\Delta 2}$

et $I_{\Delta_1} + I_{\Delta_2} = I_\Delta$

$\rightarrow$ $I_{\Delta_1} = I_{\Delta_2} = \frac{I_\Delta}{2} = \frac{1}{5} m R^2$

mais $m_1 = m_2 = \frac{m}{2}$ $\rightarrow$ $\boxed{I_{\Delta_1} = \frac{2}{5} m_1 R^2}$

7) $I_{Ax} = 0$ $\left(= \int (y^2 + z^2) dm = \underbrace{(y^2 + z^2)}_{0} \int dm \right)$

$$I_{Ay} = \int_0^{\ell} (x^2 + z^2)\, dm = \int_0^{\ell} x^2 \lambda\, dx = \lambda \left[\frac{x^3}{3} \right]_0^{\ell}$$

$m = \lambda \ell$ $\rightarrow$ $\ell = \frac{m}{\lambda}$ $\rightarrow$ $\boxed{I_{Ay} = \frac{m \ell^2}{3}}$

$$I_{Gy} = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} x^2 \lambda dx = \frac{\lambda \ell^3}{12} = \frac{m\ell^2}{12}$$

ou : Théorème de Huyghens :

$$I_{Gy} = I_{Ay} - m\, d^2(A_y, G_y)$$

$$= \frac{m\ell^2}{3} - \frac{m\ell^2}{4} \quad \rightarrow \quad \boxed{I_{Gy} = \frac{m\ell^2}{12}} = I_{Gz}$$

par symétrie.

8) $I_{Ay} = \int (x^2+z^2)\,dm = \int x^2 \lambda\, d\cancel{x}$ $(z=0)$

2 choix de coordonnées :

cartésiennes : $dr = \dfrac{dx}{\sin\alpha}$, x varie de 0 à $l \sin\alpha$

$$\rightarrow \quad I_{Ay} = \frac{\lambda}{\sin\alpha} \int_0^{l\sin\alpha} x^2 dx = \frac{1}{3}\lambda \ell^3 \sin^2\alpha = \frac{1}{3} m \ell^2 \sin^2\alpha$$

polaire : $x = r\sin\alpha$, r varie de 0 à ℓ

$$\rightarrow \quad I_{Ay} = \lambda \sin^2\alpha \int_0^{\ell} r^2 dr = \frac{1}{3}\lambda \sin^2\alpha\, \ell^3$$

$$\boxed{I_{Ay} = \frac{1}{3} m \ell^2 \sin^2\alpha}$$

Pour I_{Ax}, il suffit de remplacer $\sin\alpha$ par $\cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$

$$\rightarrow \quad \boxed{I_{Ax} = \frac{1}{3} m \ell^2 \cos^2\alpha}$$

I_{Az} est identique à I_{Ay} de l'exercice 7) : $\boxed{I_{Az} = \dfrac{m\ell^2}{3}}$

$$I_{Gy} = I_{Ay} - m\left(\frac{l\sin\alpha}{2}\right)^2 \quad \rightarrow \quad \boxed{I_{Gy} = \frac{m\ell^2\sin^2\alpha}{12}}$$

9) $$I_{Gxz} = \int y^2\,dm = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} y^2\,dy \int_{-\frac{c}{2}}^{\frac{c}{2}} dz = \frac{\rho\, a b^3 c}{12}$$

$$m = \rho\, abc \quad \rightarrow \quad \boxed{I_{Gxz} = \frac{m b^2}{12}}$$

$$\rightarrow \quad \boxed{I_{Gyz} = \frac{m a^2}{12}}$$ a joue le rôle de b ds I_{Gxz}

$$I_{Gz} = I_{Gxz} + I_{Gyz} = \boxed{\frac{1}{12} m\left(a^2 + b^2\right) = I_{Gz}}$$

$$I_{Az} = I_{Gz} + m d^2 \quad \text{avec} \quad d^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2$$

$$\rightarrow \quad \boxed{I_{Az} = \frac{m}{3}\left(a^2 + b^2\right)}$$

si $c \ll a$ et b : I_{Gz} est indépendant de c → son expression reste la m̂.

et $$I_{Gx} = I_{Gxy} + I_{Gxz} = \cancel{\frac{m c^2}{12}} + \frac{m b^2}{12}$$

$$\rightarrow \quad \boxed{I_{Gx} = \frac{m b^2}{12}}$$

$$I_{Ax} = I_{Gx} + m d^2$$

$$= \frac{m b^2}{12} + m \left[\left(\frac{b}{2}\right)^2 + \cancel{\left(\frac{c}{2}\right)^2} \right]$$

$$\boxed{I_{Ax} = \frac{5 m b^2}{12}}$$