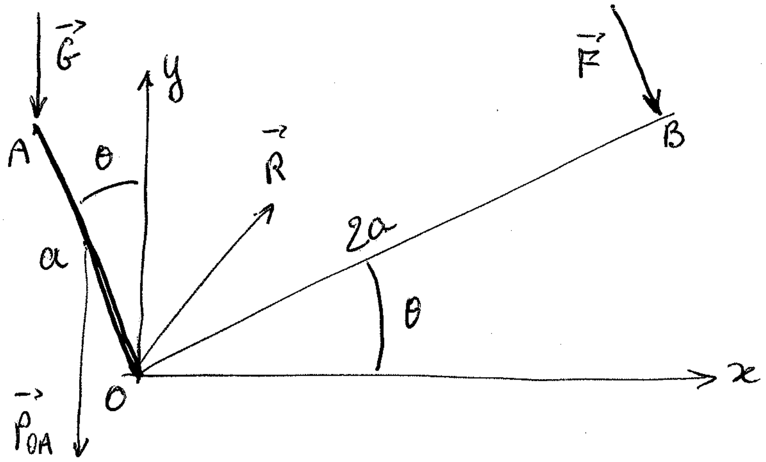
Equilibre statique d'une barre coudée



11)

$$\vec{G} = \begin{vmatrix} 0 \\ -G \\ 0 \end{vmatrix} \qquad \vec{F} = \begin{vmatrix} F\sin\theta \\ -F\cos\theta \\ 0 \end{vmatrix} \qquad \vec{P}_{OA} = \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix} \qquad \vec{R} = \begin{vmatrix} R_x \\ R_y \\ R_z \end{vmatrix}$$

équilibre statique : $\sum \vec{F}_{ext} = \vec{0}$ $\qquad \sum \overrightarrow{\mathcal{M}_O(\vec{F}_{ext})} = \vec{0}$

12) 121) On calcule la résultante des moments en O :

$$\sum \overrightarrow{\mathcal{M}_O(\vec{F}_{ext})} = \vec{OA} \wedge \vec{G} + \vec{OB} \wedge \vec{F} + \frac{\vec{OA}}{2} \wedge \vec{P}_{OA} + \cancel{\vec{OO} \wedge \vec{R}}$$

$$= \begin{vmatrix} -a\sin\theta \\ a\cos\theta \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ -G \\ 0 \end{vmatrix} + \begin{vmatrix} 2a\cos\theta \\ 2a\sin\theta \\ 0 \end{vmatrix} \wedge \begin{vmatrix} F\sin\theta \\ -F\cos\theta \\ 0 \end{vmatrix} + \begin{vmatrix} -\frac{a}{2}\sin\theta \\ \frac{a}{2}\cos\theta \\ 0 \end{vmatrix} \wedge \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix}$$

$$= \left[aG\sin\theta - 2aF\cos^2\theta - 2aF\sin^2\theta + \frac{amg}{2}\sin\theta\right]\vec{e_z} = 0$$

$$\Rightarrow \quad aG\sin\theta - 2aF + \frac{amg}{2}\sin\theta = 0$$

$$\left(G + \frac{mg}{2}\right)\sin\theta = 2F \quad \rightarrow \quad \boxed{\theta = \arcsin\left[\frac{2F}{G + \frac{mg}{2}}\right]}$$

Mouvement Circulaire uniformément accéléré

Coordonnées polaire : $\vec{OM} = r\vec{e_r}$

$$\vec{v} = \begin{vmatrix} \dot{r} \\ r\dot{\theta} \end{vmatrix} \qquad \vec{a} = \begin{vmatrix} \ddot{r} - r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{vmatrix}$$

21) ici $R = cte$ et $\ddot{\theta} = cte \Rightarrow r = R,\ \dot{r} = 0,\ \ddot{r} = 0$

$$\Rightarrow \quad \vec{OM} = R\,\vec{e_r} \qquad \vec{v} = R\dot{\theta}\,\vec{e_\theta} \qquad \vec{a} = -R\dot{\theta}^2\vec{e_r} + R\ddot{\theta}\,\vec{e_\theta}$$

$\dot{\theta} = \int \ddot{\theta}\,dt = \ddot{\theta}\,t + cte$ à $t=0$; $\dot{\theta} = 0 \rightarrow cte = 0$

$$\Rightarrow \boxed{\dot{\theta} = \ddot{\theta}\,t}$$

$\theta = \int \dot{\theta}\,dt = \ddot{\theta}\frac{t^2}{2} + cte$ à $t=0$; $\theta = 0 \rightarrow cte = 0$

$$\Rightarrow \theta = \frac{1}{2}\ddot{\theta}\,t$$

en B, $\theta = \pi \Rightarrow$ $\boxed{t = \left(\frac{2\pi}{\ddot{\theta}}\right)^{1/2}}$

221) $\vec{dl} = d\vec{OM} = dr\,\vec{e_r} + r\,d\vec{e_r}$

222) $\vec{a} = d^2\vec{OM}/dt^2 \ldots$ (cours et plus haut)

223) Principe fondamental de la dynamique : $\vec{F} = m\vec{a}$

de façon générale en base polaire : $\vec{F} = f_r\,\vec{e_r} + f_\theta\,\vec{e_\theta}$

avec $\vec{a} = -R\dot{\theta}^2\vec{e_r} + R\ddot{\theta}\,\vec{e_\theta}$: $\vec{F} = m\left(-R\dot{\theta}^2\vec{e_r} + R\ddot{\theta}\,\vec{e_\theta}\right)$

identificat° :
$$\boxed{\begin{aligned} f_r &= -mR\dot{\theta}^2 \\ f_\theta &= mR\ddot{\theta} \end{aligned}}$$

23) On peut ici utiliser 2 méthode :

a) définition du travail élémentaire :

$$\delta W = \vec{F}\cdot\vec{dl} = \vec{F}\cdot d\vec{OM}$$

$$d\vec{OM} = d(R\,\vec{e_r}) = R\,d\vec{e_r} = R\,d\theta\,\vec{e_\theta} \qquad \left(\vec{e_\theta} = \frac{d\vec{e_r}}{d\theta}\right)$$

$$\vec{F} = -mR\dot{\theta}^2\,\vec{e_r} + mR\ddot{\theta}\,\vec{e_\theta} \qquad \text{(exercice 12 :)}$$

$$\to \delta W = \left(-mR\dot{\theta}^2\vec{e_r} + mR\ddot{\theta}\vec{e_\theta}\right).\,R\,d\theta\,\vec{e_\theta}$$

$$= mR^2\ddot{\theta}\,d\theta$$

$\vec{F}$ va de $A(\theta=0)$ à $B(\theta=\pi)$: $W_{A\to B} = \int_0^\pi \delta W$

$$W_{A\to B} = \int_0^\pi mR^2\ddot{\theta}\,d\theta = mR^2\ddot{\theta}\,[\theta]_0^\pi = m\pi R^2\ddot{\theta}$$

$$\boxed{W_{A\to B} = m\pi R^2\ddot{\theta}}$$

b) Théorème de l'énergie cinétique :

$$W_{A\to B} = \frac{1}{2}m v_B^2 - \frac{1}{2}m v_A^2$$

$$v_A = 0 \quad \text{et} \quad v_B = R\dot{\theta}_B = R\ddot{\theta}\,t_B = R\ddot{\theta}\left(\frac{2\pi}{\ddot{\theta}}\right)^{1/2}$$

$$\Rightarrow W_{A\to B} = \frac{1}{2}m\left(R\ddot{\theta}\right)^2.\,\frac{2\pi}{\ddot{\theta}} = \underline{m\pi R^2\ddot{\theta}}$$

24) $$\vec{\mathcal{M}}_0(\vec{F}(M)) = \vec{OM}\wedge\vec{F} = \begin{vmatrix} R \\ \\ 0 \end{vmatrix} \wedge \begin{vmatrix} -mR\dot{\theta}^2 \\ \\ mR\ddot{\theta} \end{vmatrix} = mR^2\ddot{\theta}\,\vec{e_z}$$

25) $$\vec{\sigma}_0 = \vec{OM}\wedge\vec{p} = \begin{vmatrix} R \\ 0 \end{vmatrix} \wedge m \begin{vmatrix} R \\ R\dot{\theta} \end{vmatrix} = mR^2\dot{\theta}\,\vec{e_z}$$

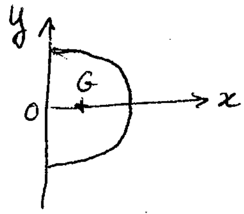
Théorème du moment cinétique :

$$\frac{d\vec{\sigma}_0}{dt} = m R^2 \frac{d\dot{\theta}}{dt} \vec{e_z} = m R^2 \ddot{\theta} \vec{e_z} = \vec{\mathcal{M}}_0(\vec{F}(M))$$

cqfd

Centre de Masse et Moment d'inertie

31) Théorème de Guldin : $y_G = 0$ (symétrie $\%$ plan (xOz))



$$2\pi S x_G = V$$

$$S = 1/2\pi R^2 \qquad V = \frac{4}{3}\pi R^3 \quad \to \quad \frac{\cancel{2\pi}}{\cancel{2}}\cancel{\pi}\cancel{R^2} x_G = \frac{4}{3}\cancel{\pi} R^{\cancel{3}}$$

$$\Rightarrow \quad \boxed{x_G = \frac{4R}{3\pi}}$$

32)

$$I_{Gz} = I_{Oz} - m\, d^2(Gz, Oz)$$

$$d(Gz, Oz) = \frac{4R}{3\pi}$$

$$I_{Oz} = \int d^2\, dm \qquad \begin{aligned} d &= r \\ dm &= \sigma r\, d\theta\, dr \end{aligned}$$

$$I_{Oz} = \sigma \int r^3\, d\theta\, dr = \sigma \int_0^R r^3 dr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \sigma \frac{R^4}{4} \cdot \pi$$

$$m = \sigma \frac{\pi R^2}{2} \quad \to \quad \sigma = \frac{2m}{\pi R^2}$$

$$I_{Oz} = \frac{2m}{\cancel{\pi} R^2} \cdot \frac{R^4 \cancel{\pi}}{4} = \frac{mR^2}{2}$$

$$I_{Gz} = \frac{mR^2}{2} - m\left(\frac{4R}{3\pi}\right)^2 = mR^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)$$

$$\boxed{I_{Gz} = mR^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)}$$

Oscillateur Amorti

41) On avait le choix !

$$\sum \mathcal{M}_{Oz} = I_{Oz}\ddot{\theta} \ ; \qquad \frac{dE_m}{dt} = \mathcal{P}_{non\ cons} \ ; \qquad \frac{dE_c}{dt} = \mathcal{P}(\vec{F})$$

Forces : poids $\vec{P}$ appliqué en G

amortissement $\Gamma_{Oz} = -\alpha\dot{\theta}$

<u>$\sum \mathcal{M}_{Oz} = I_{Oz}\ddot{\theta}$</u> : $\overrightarrow{\mathcal{M}_O}(\vec{P}) = \vec{OG} \wedge \vec{P} = -mgb\sin\theta\,\vec{e_z}$

$\rightarrow \mathcal{M}_{Oz}(\vec{P}) = (\vec{OG} \wedge \vec{P}) \cdot \vec{e_z} = -mgb\sin\theta$

$\rightarrow I_{Oz}\ddot{\theta} = -mgb\sin\theta - \alpha\dot{\theta}$ (avec $\sin\theta \simeq \theta$)

$$\rightarrow \boxed{amR^2\ddot{\theta} + \alpha\dot{\theta} + mgb\theta = 0}$$

<u>$\dfrac{dE_m}{dt} = \mathcal{P}_{n.cons.}$</u> :

$E_c = \frac{1}{2} I_{Oz}\dot{\theta}^2 \qquad \dfrac{dE_c}{dt} = I_{Oz}\dot{\theta}\ddot{\theta}$

$E_p = mg(R - b\cos\theta) \qquad \dfrac{dE_p}{dt} = mgb\dot{\theta}\sin\theta$

$\mathcal{P}(\Gamma_{Oz}) = -\alpha\dot{\theta}^2$

$$amR^2\dot{\theta}\ddot{\theta} + mgb\dot{\theta}\sin\theta + \alpha\dot{\theta}^2 = 0$$

$$amR^2\ddot{\theta} + \alpha\dot{\theta} + mgb\theta = 0$$

$$\underline{\frac{dE_c}{dt} = \mathcal{P}(\vec{F}) :}$$

$$\frac{dE_c}{dt} = \mathcal{P}(\vec{P}) + \mathcal{P}(M_{Oz})$$

$$E_c = \frac{1}{2} I_{Oz} \dot{\theta}^2 \qquad \frac{dE_c}{dt} = I_{Oz} \dot{\theta} \ddot{\theta}$$

$$\mathcal{P}(\vec{P}) = m\vec{g} \cdot \vec{v}(G)$$

$$\vec{OG} = \begin{vmatrix} b\sin\theta \\ -b\cos\theta \\ 0 \end{vmatrix} \quad \rightarrow \quad \mathcal{P}(\vec{P}) = \begin{vmatrix} 0 \\ -mg \\ 0 \end{vmatrix} \cdot \begin{vmatrix} b\cos\theta \\ b\sin\theta \\ 0 \end{vmatrix}$$

$$\vec{v}(G) = \frac{d\vec{OG}}{dt} = \dot{\theta} \begin{vmatrix} b\cos\theta \\ b\sin\theta \\ 0 \end{vmatrix} \qquad = -mgb\sin\theta\,\dot{\theta}$$

$$\rightarrow \quad I_{Oz} \dot{\theta} \ddot{\theta} = -mgb\dot{\theta}\sin\theta - \alpha\dot{\theta}^2 \qquad (\sin\theta \simeq \theta)$$

$$\underline{amR^2 \ddot{\theta} + mgb\,\theta + \alpha\dot{\theta} = 0}$$

42) On met l'équation sous la forme : $\ddot{\theta} + 2\xi\omega_0\dot{\theta} + \omega_0^2\theta = 0$

$$\ddot{\theta} + \frac{\alpha}{amR^2}\dot{\theta} + \frac{gb}{aR^2}\theta = 0$$

$$\rightarrow \quad \boxed{\omega_0 = \left(\frac{gb}{aR^2}\right)^{1/2}} \qquad \boxed{\xi = \frac{\alpha}{2mR\sqrt{abg}}}$$